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An Ore-type condition for arbitrarily vertex decomposable graphs

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Abstract

Let G be a graph of order n and r, $1 \leq r \leq n$, a fixed integer. G is said to be r-vertex decomposable if for each sequence $(n_1, \ldots, n_r)$ of positive integers such that $n_1 + \cdots + n_r = n$ there exists a partition $(V_1, \ldots, V_r)$ of the vertex set of G such that for each $i \in \{1, \ldots, r\}$, V_i induces a connected subgraph of G on n_i vertices. G is called arbitrarily vertex decomposable if it is r-vertex decomposable for each $r \in \{1, \ldots, n\}$.

In this paper we show that if G is a connected graph on n vertices with the independence number at most $\lceil n/2 \rceil$ and such that the degree sum of any pair of non-adjacent vertices is at least $n - 3$, then G is arbitrarily vertex decomposable or isomorphic to one of two exceptional graphs. We also exhibit the integers r for which the graphs verifying the above degree-sum condition are not r-vertex decomposable.

Keywords: Arbitrarily vertex decomposable graph; Traceable graph; Independence number; Perfect matching

1. Introduction

Let $G = (V, E)$ be a graph of order n. A sequence $\tau = (n_1, \ldots, n_k)$ of positive integers is called *admissible for G* if it adds up to n. If $\tau = (n_1, \ldots, n_k)$ is an admissible sequence for G and there exists a partition $(V_1, \ldots, V_k)$ of the vertex set V such that for each $i \in \{1, \ldots, k\}$, $|V_i| = n_i$ and a subgraph induced by V_i is connected, then τ is called *realizable in G* and the sequence $(V_1, \ldots, V_k)$ is said to be *a G-realization of* τ or *a realization of* τ *in* G. A graph G is *arbitrarily vertex decomposable* (*avd* for short) if for each admissible sequence τ for G there exists a G-realization of τ. Similarly, G is *r-vertex decomposable* if each admissible sequence $(n_1, \ldots, n_r)$ of r components is realizable in G.

It is clear that each avd graph admits a perfect matching or a matching that omits exactly one vertex. Note also that if G_1 is a spanning subgraph of a graph G_2 and G_1 is avd, then so is G_2.

The problem of describing avd trees has been treated in several papers. It is worth pointing out that the investigation of trees is motivated by the fact that a connected graph is avd if one of its spanning trees is avd.

In [8] Horňák and Woźniak conjectured that if T is a tree with maximum degree $\Delta(T)$ at least five, then T is not avd. This conjecture was proved by Barth and Fournier [2]. The first result characterizing non-trivially avd trees (i.e., caterpillars with three leaves) was found by Barth et al. [1] and, independently, by Horňák and Woźniak [7] (see

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Section 3). In [1,2] Barth et al. and Barth and Fournier studied a family of trees each of them being homeomorphic to $K_{1,3}$ or $K_{1,4}$ (they call them tripods or 4-pods) and showed that determining if such a tree is avd can be done using a polynomial algorithm. In [4] Cichacz et al. gave a complete characterization of arbitrarily vertex decomposable caterpillars with four leaves. They also described two infinite families of arbitrarily vertex decomposable trees with maximum degree three or four.

There are also some results on avd graphs which admits cycles. Győri [5] and, independently, Lovász [10] proved that every k-connected graph is k-vertex decomposable. In [9] Kalinowski et al. investigated unicyclic avd graphs where the unique cycle is dominating.

However, it is obvious that each graph having a hamiltonian path (i.e., a path that contains all the vertices of the graph) is avd. Therefore, each condition implying the existence of a hamiltonian path in a graph also implies that the graph is avd. So one can try to replace some known conditions for traceability by the weaker ones implying that the graphs satisfying these conditions are avd.

Observe that any necessary condition for a graph to contain a perfect matching (or a matching that omits exactly one vertex) is a necessary condition for a graph to be arbitrarily vertex decomposable. Thus we will assume that the independence number of an n-vertex graph is at most $\lceil n/2 \rceil$.

The well-known Ore's theorem [12] implies that if G is an n-vertex graph such that the degree sum of any two non-adjacent vertices is at least $n-1$ (i.e., G satisfies the Ore-type condition with the bound $n-1$), then G has a hamiltonian path.

The aim of this paper is to show that every connected graph of order $n \geq 8$ satisfying the Ore-type condition with the bound $n-3$ is avd provided its independence number is at most $\lceil n/2 \rceil$. The main result (Theorem 2) is presented in Section 4. This is an extension of two results of [11] (Corollaries 1 and 3 of Section 5).

In Section 5 we examine the structure of graphs that satisfy an Ore-type condition and are not avd and we exhibit the admissible sequences which are not realizable in the graphs under consideration.

Notice that the problem of deciding whether a given graph is arbitrarily vertex decomposable is NP-complete [1] but we do not know if this problem is NP-complete when restricted to trees. Note also that one can find in [8] some references concerning arbitrarily edge decomposable graphs.

Another interesting problem related to a notion of avd graphs is the characterization of on-line arbitrarily vertex decomposable graphs. The complete characterization of on-line avd trees has been recently found by Horňák et al. [6].

2. Terminology and notation

In this paper, we deal with finite, simple and undirected graphs. If $G = (V, E)$ is a graph, then $V = V(G)$ is the vertex set of G, and $E = E(G)$ is the set of edges of G. By $N(x)$ we denote the set of vertices adjacent to a vertex x, and the number $d(x) = |N(x)|$ is the *degree* of x in G.

Let $T = (V, E)$ be a tree. A vertex $x \in V$ is called *primary* if $d(x) \geq 3$. A *leaf* (or a *hanging vertex*) is a vertex of degree one. A path P of T is an *arm* if one of its endvertices is a leaf in T, the other one is primary and all internal vertices of P have degree two in T. A graph T is a *star-like tree* if it is a tree homeomorphic to a star $K_{1,q}$ for some $q \geq 3$. Such a tree has one primary vertex and q arms $A_1, A_2, \ldots, A_q$. For each A_i let $a_i \geq 2$ be the order of A_i. We shall denote the above defined star-like tree by $S(a_1, \ldots, a_q)$. Notice that the order of this star-like tree is equal to $1 + \sum_{i=1}^{q}(a_i - 1)$.

Let G be a graph and let $P = x_1, x_2, \ldots, x_r$ be a path of G with a natural orientation (from x_1 to x_r). For a vertex a of $P - x_r$ we denote by a^+ the successor of a on P and for $a \in V(P) \setminus \{x_1\}$ we denote by a^- its predecessor on P. We write a^{+2} for $(a^+)^+$, a^{-2} for $(a^-)^-$, and, by induction, a^{+k} for $(a^{+(k-1)})^+$ and a^{-k} for $(a^{-(k-1)})^-$. Let $A = \{a_1, a_2, \ldots, a_p\} \subseteq V(P) \setminus \{x_r\}$. We shall write A^+ for the set $\{a_1^+, a_2^+, \ldots, a_p^+\}$. We define the set A^- in a similar way.

Let $a = x_i$ and $b = x_j$ be two vertices of P such that $i < j$. By aPb we denote the path $x_i, x_{i+1}, \ldots, x_j$. It will be called *segment* of P from a to b. If $x \notin V(P)$ we write $N_P(x)$ for the set of neighbors of x on P and we denote by $d_P(x)$ the number $|N_P(x)|$.

We denote by $\alpha(G)$ the *independence number* of a graph G, i.e., the maximum number of pairwise non-adjacent vertices in G.

The *join* of two vertex-disjoint graphs G and H is the graph denoted by $G \vee H$ obtained from $G \cup H$ by adding all edges between G and H.

Let G be a graph of order n. Define

$$\sigma_2(G) := \min\{d(x) + d(y) \mid x, y \text{ are non-adjacent vertices in } G\}$$

if G is not a complete graph, and $\sigma_2(G) = \infty$ otherwise. The well-known Ore's theorem [12] states that every n-vertex graph G with $\sigma_2(G) \geq n \geq 3$ is hamiltonian. This implies at once that if the order of a graph G is n and $\sigma_2(G) \geq n - 1$, then G contains a hamiltonian path, so it is also avd.

A graph containing a hamiltonian path is often called *traceable*.

3. Preparatory results

The first result characterizing avd star-like trees (i.e., caterpillars with one single leg) was found by Barth et al. [1] and, independently, by Horňák and Woźniak [7].

Proposition 1. *A star-like tree $S(2, a, b)$ is avd if and only if the integers a and $n = a + b$ are coprime. Moreover, each admissible and non-realizable sequence in $S(2, a, b)$ is of the form $(d, d, \ldots, d)$, where $a \equiv n \equiv 0 \pmod d$ and $d > 1$.*

The next proposition was presented in [11]. However, for the sake of completeness we give here a short proof of this result.

Proposition 2. *Let G be the graph of order $n \geq 4$ obtained by taking a path $P = x_1, \ldots, x_{n-1}$, a single vertex x and by adding the edges $xx_{i_1}, xx_{i_2}, \ldots, xx_{i_p}$, where $1 < i_1 < \cdots < i_p < n - 1$ and $p \geq 1$. Then G is not avd if and only if there are integers $d > 1$, λ, $\lambda_1, \lambda_2, \ldots, \lambda_p$ such that $n = \lambda d$ and $i_j = \lambda_j d$ for $j = 1, \ldots, p$. Moreover, each admissible and non-realizable sequence in G is of the form $(d, d, \ldots, d)$, where $i_j \equiv n \equiv 0 \pmod d$ $(j = 1, \ldots, p)$ and $d > 1$.*

Proof. Suppose that the integers $d > 1$, λ, $\lambda_1, \lambda_2, \ldots, \lambda_p$ satisfy the conditions $n = \lambda d$ and $i_j = \lambda_j d$ for $j = 1, \ldots, p$ and consider the admissible sequence $\tau = (\underbrace{d, \ldots, d}_{\lambda})$ for G. Observe that if G' is a connected subgraph of G of order d which contains the vertex x, then the connected component of $G - V(G')$ containing the vertex x_1 is a path P' such that d does not divide the order of P'. Thus, τ is not realizable in G. Conversely, if $\tau = (n_1, n_2, \ldots, n_\lambda)$ is an admissible sequence for G that is not realizable in G, then τ is also not realizable in the caterpillar $S(2, i_1, n - i_1)$. By Proposition 1, there are two integers $d > 1$ and λ_1 such that $n_1 = n_2 = \cdots = n_\lambda = d$ and $i_1 = \lambda_1 d$. The sequence τ cannot be realizable in the caterpillar $S(2, i_2, n - i_2)$, therefore, again by Proposition 1, $i_2 = \lambda_2 d$ for some integer λ_2. Repeating the same argument we prove that the conditions of the proposition hold. ■

In the proofs of the main results of this paper we will need the following results. The first one is due to Ore [12].

Proposition 3. *Let G be a graph of order $n \geq 3$ and $x_1, \ldots, x_n$ a hamiltonian path in G such that $d(x_1) + d(x_n) \geq n$ and $x_1 x_n \notin E(G)$. Then G is hamiltonian.*

Proof. Let $A = N(x_1)$ and $B = N(x_n)$. Suppose G is not hamiltonian. If for some j, $x_j \in A \cap B^+$, then, because $x_1 x_n \notin E(G)$, $j \geq 3$, $x_1 x_j \in E(G)$ and $x_{j-1} x_n \in E(G)$, so $x_j, x_1, x_2, \ldots, x_{j-1}, x_n, x_{n-1}, \ldots, x_j$ is a hamiltonian cycle, a contradiction. Thus $A \cap B^+ = \emptyset$ and $A \cup B^+ \subseteq \{x_2, \ldots, x_n\}$, so $d(x_1) + d(x_n) = |A| + |B| = |A| + |B^+| \leq n - 1$, a final contradiction. ■

The second result is attributed to Pósa [13] (cf. [3]).

Theorem 1. *Let G be a connected graph of order $n \geq 3$ such that*

$$\sigma_2(G) \geq d.$$

If $d < n$ then G contains a path of length d and if $d \geq n$, then G is hamiltonian.

Proof. Let P be a longest path in G and l be the length of P. If $l < n - 1$ and $l < d$, then we can apply Proposition 3 to the subgraph induced by $V(P)$ and find a cycle C of length $l + 1$ with $V(C) = V(P)$. Since G is connected, it also contains a path of length $l + 1$, a contradiction. For $l = n - 1$ the assertion is true by Proposition 3. ■

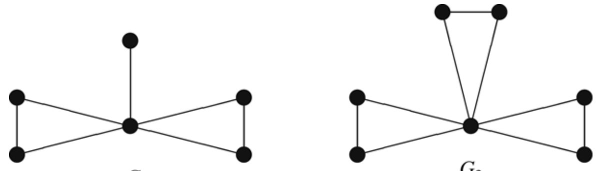

Fig. 1. Two non-avd graphs.

4. Main result

Let G_1 be the join $K_1 \vee (K_1 \cup 2K_2)$, where $2K_2$ denotes two disjoint copies of K_2 (see Fig. 1). This graph is not avd because the sequence $(3, 3)$ is not realizable in G_1. It is easy to check that $\sigma_2(G_1) = n - 3 = 3$ and $\alpha(G_1) = 3 = \lceil n/2 \rceil$, where $n = 6$ is the order of G_1. Consider now the graph $G_2 = K_1 \vee 3K_2$ (see Fig. 1). It can be easily seen that the sequences $(3, 3, 1)$ and $(4, 3)$ are not realizable in G_2, but $\sigma_2(G_2) = n - 3 = 4$ and $\alpha(G_2) = 3 < \lceil n/2 \rceil$, where $n = 7$ is the order of G_2.

Theorem 2. *Let G be a connected graph of order n such that $\sigma_2(G) \geq n - 3$, $\alpha(G)$ is at most $\lceil n/2 \rceil$ and G is isomorphic neither to G_1 nor to G_2. Then G is avd.*

Proof. Suppose G is not avd and satisfies the hypothesis of our theorem. Then G is not traceable, so $n \geq 4$, and by Theorem 1, there exists in G a path of length at least $n - 3$.

Case 1: The length of a longest path is $n - 3$.

Let $P = x_1, x_2, \ldots, x_{n-2}$ be such a path and let x and y be two vertices outside P such that $d_P(x) \geq d_P(y)$. Denote by $A = N_P(x)$ the set of neighbors of x on P and let $p := d_P(x) = |A|$.

Case 1.1: x and y are not adjacent.

Hence $d_P(x) = d(x) \geq (n - 3)/2$ and, since G is connected and the length of the longest path equals $n - 3$, we have $p \geq 1$, $x_1x \notin E$, $x_{n-2}x \notin E$ and $x_1x_{n-2} \notin E(G)$. Furthermore, there is at least one vertex between any two consecutive neighbors of x on P, i.e., $A \cap A^+ = \emptyset$ and $A \cup A^+ \subseteq \{x_2, x_3, \ldots, x_{n-2}\}$. It follows that $d(x) = |A| \leq (n - 3)/2$, so $d(x) = (n - 3)/2$, $n \geq 5$ is odd and $A = \{x_2, x_4, \ldots, x_{n-3}\}$.

Since x and y are not adjacent, we have $d(y) \geq (n - 3)/2$ and using the similar argument as above we can show that $d(y) = (n - 3)/2$ and $N(y) = A$. Observe now that $x_1u \notin E(G)$ for each $u \in A^+$, for otherwise $x, u^-, u^{-2}, \ldots, x_1, u, \ldots, x_{n-2}$ is a path of length $n - 2$ in G, a contradiction. Using the similar argument we can show that $x_{n-2}u \notin E(G)$ for each $u \in A^+ \setminus \{x_{n-2}\}$. It is obvious that any edge of the form $x_{2i-1}x_{2j-1}$ would create a path of length at least $n - 2$ in G, so the set $\{x, y, x_1, x_3, \ldots, x_{n-4}, x_{n-2}\}$ of $(n + 3)/2$ vertices is independent and we obtain a contradiction.

Case 1.2: x and y are adjacent.

Obviously, the vertices $x_1, x_2, x_{n-3}, x_{n-2}$ do not belong to $N(x) \cup N(y)$, since otherwise G would contain a path of length $n - 2$. We have by assumption $p + 1 + d(x_1) = d(x) + d(x_1) \geq n - 3$, thus $d(x_1) \geq n - 4 - p$. On the other hand we can show as in the previous case that if $u \in A^+$ then $x_1u \notin E$, and, because $x_1x_{n-2} \notin E$, $xx_{n-2} \notin E$ and $xx_{n-3} \notin E$, we have $A^+ \subseteq \{x_4, \ldots, x_{n-3}\}$ and $d(x_1) \leq n - 4 - p$. It means that x_1 is adjacent to each vertex of $V(P) \setminus (A^+ \cup \{x_{n-2}\})$. If $x_rx \in E(G)$ and $r < n-4$, then x_r^{+2} is adjacent to x_1 and it is easy to check that G contains a path of length $n-2$, a contradiction. Hence x_{n-4} is the only neighbor of x. Thus, $p = 1$, and, by symmetry, $xx_3 \in E$, so $n - 4 = 3$, x_1 and x_5 are adjacent to x_3. Thus, $n = 7$, $d(x_1) = d(x_5) = n - p - 4 = 2$ and $\{x_1, x_5\} \subset N(x_3)$. Therefore, since x_1 and y are not adjacent and $d(x_1) = 2$, we have $d(y) = 2$ and $N(y) = \{x, x_3\}$. Since x_2 and x_4 cannot be adjacent, G is isomorphic to G_2, which contradicts our assumption.

Case 2: The length of a longest path equals $n - 2$.

Let $Q = x_1, x_2, \ldots, x_{n-1}$ be a path of length $n - 2$ and x the unique vertex outside Q. Let $A = N(x) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_p}\}$, $1 \leq i_1 < \cdots \leq i_p \leq n - 1$, be the set of neighbors of x. Since G is connected and non-traceable, we have $p \geq 1$, $i_1 > 1$, $i_p < n - 1$ and $x_1x_{n-1} \notin E(G)$. By Proposition 2, there are integers $d > 1, \lambda, \lambda_1, \lambda_2, \ldots, \lambda_p$ such that $n = \lambda d$ and $i_j = \lambda_j d$ for $j = 1, \ldots, p$. Hence, there is at least one vertex between any two consecutive neighbors of x on Q.

Since $x_1x \notin E(G)$ and $x_{n-1}x \notin E(G)$, it follows by assumption that $d(x_1) \geq n-3-p$ and $d(x_{n-1}) \geq n-3-p$. We can show as in the previous case that if $u \in A^+$, then $x_1u \notin E(G)$. Therefore, $d(x_1) \leq n-2-p$, hence $d(x_1) \in \{n-3-p, n-2-p\}$.

Case 2.1: $x_{n-2}x \in E$, i.e., $i_p = n-2$.

Thus, using Proposition 2, $d = 2$, n is even and $\tau = (2, 2, \ldots, 2)$ is the only non-realizable sequence for G. Moreover, every path $x_{i_j}Qx_{i_{j+1}}$ is of even length, i.e., it contains an odd number of vertices.

Case 2.1.1: There is some integer s such that $|V(x_{i_s}Qx_{i_{s+1}})| \geq 5$.

Set $u = x^+_{i_s}$ and $v = x^-_{i_{s+1}}$. Notice that $x_{n-1}u \notin E$ and $x_1v \notin E$ because G is not traceable. Thus $N(x_{n-1}) \subseteq V(Q) \setminus (\{x_{n-1}, u\} \cup A^-)$ and $N(x_1) \subseteq V(Q) \setminus (\{x_1, v\} \cup A^+)$, so $d(x_{n-1}) \leq n-3-p$ and $d(x_1) \leq n-3-p$, therefore $d(x_1) = d(x_{n-1}) = n-3-p$. If $i_1 \geq 4$ then $x_1 \notin A^-$, so $d(x_{n-1}) \leq n-1-3-p$ and we obtain a contradiction. Therefore, $xx_2 \in E$. Similarly, if for some integer $q \neq s$ we have $|V(x_{i_q}Qx_{i_{q+1}})| \geq 5$, then also $d(x_{n-1}) \leq n-4-p$, and we get a contradiction. Hence, s is the unique integer j such that $|V(x_{i_j}Qx_{i_{j+1}})| \geq 5$. Now, if $|V(x_{i_s}Qx_{i_{s+1}})| > 5$, all the vertices of the path u^+Qv^- are adjacent to x_1 and x_{n-1}, so $x_1u^{+3} \in E(G)$ and $x_{n-1}u^{+2} \in E(G)$. Then $C = x_1, u^{+3}, u^{+4}, \ldots, x_{n-1}, u^{+2}, u^+, \ldots, x_1$ is a cycle with $V(C) = V(Q)$. Hence G is traceable which contradicts our assumption. Suppose then $|V(x_{i_s}Qx_{i_{s+1}})| = 5$. If $uv \notin E$ then the set $\{x_1, v, x\} \cup A^+$ of $(n+2)/2$ vertices is independent, a contradiction. Assume that u and v are adjacent. Then the vertex $u^+ = v^-$ is connected to both x_1 and x_{n-1} and it can be easily seen that $G - \{u, v, x, x_{i_s}, x_1, u^+\}$ is the vertex-disjoint union of two traceable subgraphs of even order (possibly one of them is empty), thus G admits a perfect matching. But we have assumed that $\tau = (2, 2, \ldots, 2)$ is non-realizable sequence for G, a contradiction.

Case 2.1.2: Every path $x_{i_j}Qx_{i_{j+1}}$ contains exactly three vertices.

First suppose that $i_1 = 4$. Clearly, $N(x_{n-1}) \subseteq \{x_2, \ldots, x_{n-2}\} \setminus A^-$, so $d(x_{n-1}) = n-p-3$ and $x_2x_{n-1} \in E$. Now, if $x_1x_3 \in E$, then G contains a cycle $x_1, x_3, x_4, \ldots, x_{n-1}, x_2, x_1$ and G is traceable, a contradiction. Therefore, $A^- \cup \{x_1, x_{n-1}, x\}$ is an independent set of cardinality $(n+2)/2$ and we get a contradiction. Notice, that the same set is independent if $i_1 = 2$. Suppose then $i_1 \geq 6$. It follows that $x_{n-1}x_2 \in E(G)$ and $x_{n-1}x_4 \in E(G)$, because $d(x_{n-1}) = n-p-3$ and x_{n-1} is adjacent to each vertex of $V(Q) \setminus (A^- \cup \{x_1, x_{n-1}\})$. Now, if $x_1x_3 \in E(G)$ or $x_1x_5 \in E(G)$, then we can easily find a cycle C with $V(C) = V(Q)$. Hence G is traceable, a contradiction. So $N(x_1) \subseteq V(Q) \setminus (A^+ \cup \{x_1, x_3, x_5\})$ and $d(x_1) \leq n-p-4$, again a contradiction.

Case 2.2: $i_p \leq n-3$.

By the same argument as in previous cases, $d(x_1) = n-3-p$. If $d = 2$, then we can assume $x_2x \notin E(G)$ (and also $x_3x \notin E(G)$), for otherwise we have the situation described in Case 2.1. Hence, $N(x_{n-1}) \subseteq V(Q) \setminus (A^- \cup \{x_{n-1}, x_1\})$ and $d(x_{n-1}) = n-3-p$, whence $x_{n-1}x_2 \in E(G)$ and $x_1x_3 \in E(G)$, and we can easily find a cycle with $V(C) = V(Q)$. It follows that G is traceable, a contradiction. Therefore, $d \geq 3$. By Proposition 2, there are at least two vertices between any two consecutive neighbors of x on Q. It follows that for $p \geq 2$, x_1 is not adjacent to $x^-_{i_2}$ (otherwise G would have a hamiltonian path: $x^+_{i_1}, x^{+2}_{i_1}, \ldots, x^-_{i_2}, x_1, \ldots, x_{i_1}, x, x_{i_2}, \ldots, x_{n-1}$), so $N(x_1) \subseteq V(Q) \setminus (A^+ \cup \{x_1, x^-_{i_2}, x_{n-1}\})$ and $d(x_1) \leq n-4-p$, a contradiction. Thus $p = 1$, $d(x_1) = d(x_{n-1}) = n-4$, so, if $n \geq 7$, then $d(x_1) + d(x_{n-1}) = 2(n-4) \geq n-1$ and by Proposition 3 there is a cycle C with $V(C) = V(Q)$. Hence G is traceable, a contradiction. It follows from Proposition 2 that $n = 6$ and $d = 3$, furthermore, since $d(x_1) = d(x_5) = 2$, we have $x_1x_3 \in E$ and $x_5x_3 \in E$. Clearly, x_2 and x_4 are not adjacent, so G is isomorphic to G_1 and we get a contradiction. ■

5. Conclusions

Corollary 1. *Let G be a 2-connected graph of order n with $\sigma_2(G) \geq n-3$. Then*

G is avd,

or $n \geq 7$ is odd and $K_{\frac{n+3}{2}, \frac{n-3}{2}} \subseteq G \subseteq \overline{K}_{\frac{n+3}{2}} \vee K_{\frac{n-3}{2}}$,

or $n \geq 6$ is even, $K_{\frac{n+2}{2}, \frac{n-2}{2}} \subseteq G \subseteq \overline{K}_{\frac{n+2}{2}} \vee K_{\frac{n-2}{2}}$,

or $n \geq 8$ is even, $K_{\frac{n+2}{2}, \frac{n-2}{2}} - e \subseteq G \subseteq (\overline{K}_{\frac{n+2}{2}} \vee K_{\frac{n-2}{2}}) - e$, where e is an arbitrary edge of $K_{\frac{n+2}{2}, \frac{n-2}{2}}$.

Proof. First observe that the condition on $\alpha(G)$ is used in Cases 1.1 and 2.1 of the proof of Theorem 2. Since G is 2-connected, it is not isomorphic to any graph belonging to the set $\{G_1, G_2, K_{1,3}, K_{1,4}, (\overline{K}_4 \vee K_2) - e\}$, where e is any edge incident with a vertex of $V(\overline{K}_4)$. If the length of a longest path of G is $n-3$ and G is not avd we find the situation described in Case 1.1 of the proof of Theorem 2, so $n \geq 7$ is odd (for $n = 5$, G is isomorphic to $K_{1,4}$) and G contains an independent set S on $\frac{n+3}{2} \geq 5$ vertices. Because $\sigma_2(G) \geq n-3$, every vertex of S is adjacent to every vertex of $G - S$, thus G is the join $\overline{K}_{(n+3)/2} \vee H$, where H is any graph on $\frac{n-3}{2}$ vertices and the first assertion of the corollary follows.

Suppose the length of the longest path of G equals $n-2$ and consider again Case 2.1 of Theorem 2. Now n is even and G contains an independent set of $\frac{n+2}{2} \geq 4$ vertices (if $n = 4$ G is isomorphic to $K_{1,3}$ and if $n = 6$, G is isomorphic to $(\overline{K}_4 \vee K_2)$, $(\overline{K}_4 \vee K_2) - e$ or $K_{2,4}$), hence all of them except at most one are of degree $\frac{n-2}{2}$ and the only exceptional vertex is of degree $(n-4)/2$, so G is contained in the join $\overline{K}_{(n+2)/2} \vee H$, where H is an arbitrary graph on $\frac{n-2}{2}$ vertices and misses at most one edge between $\overline{K}_{(n+2)/2}$ and H. In the last case the order of the graph is at least 8. ■

Corollary 2. *If G is a 2-connected graph of order n such that $\sigma_2(G) \geq n-3$, then for every integer $k \notin \{(n-1)/2, n/2, (n+1)/2\}$ G is k-vertex decomposable. Moreover, each admissible and non-realizable sequence is of the form* $(2, 2, \dots, 2, 2, 3)$ *or* $(2, 2, \dots, 2)$ *or else* $(1, 2, 2, \dots, 2)$.

Proof. The graphs that are not avd appear in Cases 1.1 and 2.1 of the proof of Theorem 2. In the latter situation n is even and $(2, 2, \dots, 2)$ is the only sequence which is not realizable in G.

Suppose then n is odd, G is not avd and consider the admissible sequences $\tau_1 = (1, 2, 2, \dots, 2)$ and $\tau_2 = (2, 2, \dots, 2, 3)$ for G. Assume τ_1 or τ_2 is realizable in G. Then, since the vertices of a connected graph of order three can be partitioned into $K_1 \cup K_2$, there exists a partition of $V(G)$ into $\frac{n+1}{2}$ complete subgraphs consisting of $\frac{n-1}{2}$ copies of K_2's and exactly one copy of K_1. Therefore, by Corollary 1, $\frac{n+3}{2} \leq \alpha(G) \leq \theta(G) \leq \frac{n+1}{2}$, where $\theta(G)$ denotes the minimum number of complete subgraphs that partition $V(G)$, so we get a contradiction. So τ_1 and τ_2 are not realizable in G.

Now assume that $\tau = (n_1, n_2, \dots, n_k)$ is another admissible sequence for G. If $n_i \leq 2$ for $i = 1, \dots, k$ and $\tau \neq \tau_1$, then, from Corollary 1, τ is realizable in G. Consider again the Case 1.1 of Theorem 2, where x and y are two vertices outside the path $P = x_1, \dots, x_{n-2}$ of length $n-3 \geq 4$. Recall that $x_2 \in A = N(x) = N(y)$. Now the spanning subgraph of G consisting of the path P and two vertices x, y together with the edges xx_2, yx_2 is isomorphic to the star-like tree $S(2, 2, 2, b)$, where $b = n-3$. Suppose for some i, say $i = 1$, $n_i = n_1 \geq 4$. Set $V_1 = \{x, y, x_1, x_2, \dots, x_{n_1-2}\}$. Clearly, V_1 induces a connected subgraph of G and the graph $G - V_1$ contains a hamiltonian path, so it is easy to find a realization of τ in G. Suppose then $n_j \leq 3$ for all j and there is i, say $i = 1$, such that $n_i = n_1 = 3$. Now the set $V_1 = \{x, x_1, x_2\}$ induces a connected subgraph of G and, because y is adjacent to x_4 in G, $G - V_1$ has a spanning subgraph G' which is isomorphic to the star-like tree $S(2, 2, n-5)$. By Proposition 1, every admissible sequence for G' which is different from $(2, 2, \dots, 2)$ is realizable in G', thus τ is realizable in G provided $\tau \neq \tau_2$. ■

Corollary 3. *If G is a graph of order n with $\sigma_2(G) \geq n-2$, then G is avd or the union of two disjoint cliques or n is even and G satisfies $K_{\frac{n+2}{2},\frac{n-2}{2}} \subseteq G \subseteq \overline{K}_{\frac{n+2}{2}} \vee K_{\frac{n-2}{2}}$.*

Proof. If G is not connected and $\sigma_2(G) \geq n-2$, then G is the union of two disjoint cliques so G is not avd. Suppose then that G is a connected graph, $\sigma_2(G) \geq n-2$ and G is not avd. For $n \geq 5$ our Corollary follows from Corollary 1. For $n = 4$ the only non-avd graph is $K_{1,3}$, the graph described in Case 2.1.2 of the proof of Theorem 2. This is the desired conclusion. ■

Corollary 4. *If G is a connected graph of order n such that $\sigma_2(G) \geq n-2$, then G is k-vertex decomposable for any $k \neq n/2$. Moreover, the sequence* $(2, 2, \dots, 2)$ *is the unique admissible sequence for G which may be not realizable in G.*

We can formulate also an immediate corollary of Theorem 2 involving a Dirac-type condition.

Corollary 5. *If G is a connected graph on n vertices such that $\alpha(G) \leq \lceil n/2 \rceil$, $G \notin \{G_1, G_2\}$ and minimum degree $\delta(G) \geq \frac{n-3}{2}$, then G is avd.*

Consider now the join $G_3 = K_2 \vee 4K_2$. Clearly, G_3 is a 2-connected graph of order $n = 10$ such that $\sigma_2(G_3) = n - 4 = 6$, $\alpha(G_3) = 4 < \lceil n/2 \rceil$, however the sequence $(3, 3, 3, 1)$ is not realizable in G_3. For even $n \geq 8$ define $F_n = (\overline{K}_{(n-2)/2} \cup K_3) \vee K_{(n-4)/2}$. The independence number of this graph equals $n/2$, $\sigma_2(F_n) = n - 4$, F_n is 2-connected, however, by Tutte's Theorem, F_n has no perfect matching. These two examples show that if we lower the bound $n - 3$ in Theorem 2 then the structure of non-avd graphs verifying the corresponding Ore-type conditions becomes more diversified. However, we feel that if n is large such graphs are avd provided they admit a perfect matching or a quasi-perfect matching.

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